

Supersymmetric quantum mechanics on non-commutative space

P.K. Ghosh^a

Theory Division, Saha Institute of Nuclear Physics, Kolkata 700 064, India

Received: 24 February 2005 / Revised version: 21 April 2005 /
Published online: 22 June 2005 – © Springer-Verlag / Società Italiana di Fisica 2005

Abstract. We construct a supersymmetric quantum mechanics in terms of two real supercharges on non-commutative space in arbitrary dimensions. We obtain the exact eigenspectra of the two- and three-dimensional non-commutative superoscillators. We further show that a reduction in the phase space occurs for a critical surface in the space of parameters. At this critical surface, the energy spectrum of the bosonic sector is infinitely degenerate, while the degeneracy in the spectrum of the fermionic sector gets enhanced by a factor of two for each pair of reduced canonical coordinates. For the two-dimensional non-commutative “inverted superoscillator”, we find exact eigenspectra with a well-defined ground state for certain regions in the parameter space, which have no smooth limit to the ordinary commutative space.

PACS. 03.65.-w, 03.65.Fd, 11.30.Pb, 11.10.Nx

1 Introduction

Non-commutative space or the associated algebra arises in many branches of physics and mathematics [1]. The reason behind the recent upsurge in the study of field theory on non-commutative space is that such theories naturally appear in a particular low-energy limit of open string theory on D-brane backgrounds in the presence of a constant antisymmetric tensor field [2]. There are already many interesting results on different aspects of both perturbative [3] and non-perturbative [4] non-commutative field theory [5–7]. The study of supersymmetric field theories on non-commutative spaces with the standard anticommuting [8] or non-anticommuting [9,10] odd coordinates is also an active area of research.

Along the same line of development, quantum mechanics on non-commutative space has been studied extensively [11–20]. The complete eigenstates of the non-commutative oscillator [14,15,18,19] have been found analytically in two and higher dimensions. In particular, the spectrum of the non-commutative oscillator is shown to be identical to that of an anisotropic oscillator on the corresponding commutative space. The frequencies of the anisotropic oscillator are, in general, a function of the parameters appearing in the non-commutative algebra of the phase-space operators. In the appropriate commutative limit, the spectrum of an isotropic oscillator on commutative space is recovered. No other non-trivial quantum mechanical problem is known to be completely solvable on the non-commutative plane or higher dimensions.

Supersymmetric quantum mechanics on commutative space is a well developed subject [21]. The study of su-

persymmetric quantum mechanics enriched us in understanding many subtle issues of supersymmetric field theory in a much simpler way. Further, different ideas and techniques emerging from this subject can be used to solve algebraically a host of quantum mechanical problems arising in diverse branches of physics, including those found in any standard textbook on quantum mechanics. The \star -product formulation of supersymmetric quantum mechanics on commutative space is also known [22].

It is natural to enquire at this juncture about the present status of the supersymmetric quantum mechanics on non-commutative space. Recently, the Pauli equation on the non-commutative plane was shown to be supersymmetric with the gyro-magnetic ratio $g = 2$ [23]. The superoscillator on the non-commutative plane has also been constructed [24]. However, we find that a general and uniform formulation of the $\mathcal{N} = 2$ supersymmetric quantum mechanics on non-commutative space is lacking. For example, the supersymmetric constructions in [23,24] are not suitable,

- (i) for dimensions $N \geq 3$,
- (ii) for describing quantum mechanical Hamiltonian with any arbitrary superpotential and
- (iii) for accommodating more general non-commutativities among phase-space operators.

The purpose of this paper is to present a formulation of supersymmetric quantum mechanics on non-commutative space without the shortcomings discussed above. In particular, we construct supersymmetric quantum mechanics in terms of $\mathcal{N} = 2$ real supercharges on non-commutative space in arbitrary dimensions. This construction is valid for any arbitrary superpotential and accommodates a very large class of non-commutativities among phase-space op-

^a e-mail: pijush@theory.saha.ernet.in

erators. We study the superoscillator on two- and three-dimensional non-commutative spaces in some detail.

We obtain the complete spectrum of the two-dimensional superoscillator analytically. The spectrum is identical to that of an anisotropic superoscillator on the commutative plane. The frequencies of the anisotropic superoscillators are functions of the parameters appearing in the non-commutative algebra of the phase-space operators. In the commutative limit, the isotropic superoscillator is recovered. Further, we determine the spectrum of the “inverted superoscillator” on the non-commutative plane with a well-defined ground state for certain regions in the parameter space. There is no smooth limit to the commutative space from these regions of the parameter space. Although the Hamiltonian of the “inverted superoscillator” on the non-commutative plane is non-hermitian and not semi-positive definite, the spectrum obtained is indeed real and positive.

A general analysis of the three-dimensional non-commutative superoscillator is messy and complicated. So, we present the energy spectrum for a few specific choices of the parameters. For all these cases, the energy spectrum is identical to that of an anisotropic superoscillator on three-dimensional commutative space. The commutative limit for all these cases is smooth and reproduces the well known results for usual isotropic superoscillator in three dimensions.

We show that for both two- and three-dimensional non-commutative superoscillators, a reduction in the phase space occurs for a critical surface in the parameter space. Consequently, the energy spectrum in the bosonic sector becomes infinitely degenerate, while the degeneracy in the fermionic sector gets enhanced by a factor of two for each pair of reduced canonical coordinates. The disparity in the enhancement of degeneracy between the bosonic and the fermionic sector can be attributed to the fact that a bosonic quantum number can take any non-negative integer values, while a fermionic quantum number can take only two values.

The plan of this paper is the following. We introduce the non-commutative algebra among different phase-space operators in the next section. We also construct $\mathcal{N} = 2$ supersymmetric quantum mechanics on non-commutative space for an arbitrary superpotential. The super-Hamiltonian for the particular case of superoscillator in N dimensions is obtained. In Sect. 3, we study the two-dimensional superoscillator in detail and obtain its energy spectrum. Different limiting cases including the “inverted superoscillator” is discussed in this section. The superoscillator in three dimensions is discussed in Sect. 4. Finally, we conclude in Sect. 5. In Appendix A, the procedure for diagonalizing the three-dimensional bosonic Hamiltonian is discussed. In Appendix B, the matrix representation of the fermionic Hamiltonian for $N = 3$ is given.

2 General formulation

Consider the non-commutative algebra,

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\theta_{ij}, & [\hat{p}_i, \hat{p}_j] &= iB_{ij}, \\ [\hat{x}_i, \hat{p}_j] &= i\delta_{ij} + i(1 - \delta_{ij}) C_{ij}, & i, j &= 1, 2, \dots, N, \end{aligned} \quad (1)$$

where θ_{ij} and B_{ij} are real, antisymmetric matrices and are independent of the hermitian operators \hat{x}_i, \hat{p}_i . The diagonal elements of the matrix C_{ij} are taken to be zero and its off-diagonal elements do not depend on the space and momentum coordinates. We introduce $2N$ elements ξ_i satisfying the following real Clifford algebra:

$$\{\xi_i, \xi_j\} = 2g_{ij}, \quad g_{ij} = \begin{pmatrix} -\mathcal{I} & 0 \\ 0 & \mathcal{I} \end{pmatrix}, \quad (2)$$

where \mathcal{I} is an $N \times N$ identity matrix. The signature of the metric g_{ij} is such that the square of the ξ_i 's is equal to 1 or -1 depending on whether $i > N$ or $i \leq N$, respectively. In particular,

$$\xi_{N+i}^2 = -\xi_i^2 = 1, \quad i = 1, 2, \dots, N. \quad (3)$$

In this paper, we use a particular matrix representation of the Clifford algebra so that the following additional relations are also satisfied,

$$\xi_i^\dagger = -\xi_i, \quad \xi_{N+i}^\dagger = \xi_{N+i}, \quad (4)$$

where X^\dagger denotes the hermitian adjoint of X . Such a matrix representation of the Clifford algebra is required to construct hermitian Hamiltonian within our approach. We further introduce the hermitian operator γ_5 ,

$$\gamma_5 = \xi_1 \xi_2 \dots \xi_{2N-1} \xi_{2N}, \quad (5)$$

which anticommutes with all the ξ_i 's and $\gamma_5^2 = 1$. The elements of the Clifford algebra ξ_i, ξ_{N+i} commute with the non-commutative bosonic variables \hat{x}_i and \hat{p}_i ,

$$[\hat{x}_i, \xi_j] = [\hat{p}_i, \xi_j] = 0, \quad [\hat{x}_i, \xi_{N+j}] = [\hat{p}_i, \xi_{N+j}] = 0, \quad \forall i, j. \quad (6)$$

It naturally follows that γ_5 also commutes with all the bosonic coordinates.

We now introduce the supercharges Q_1 and Q_2 [25,26],

$$Q_1 = \frac{1}{\sqrt{2}} \sum_{i=1}^N \left[-i \xi_i \hat{p}_i + \xi_{N+i} \hat{W}_i \right], \quad Q_2 = -i\gamma_5 Q_1. \quad (7)$$

The superpotential \hat{W}_i are real functions of the non-commutative coordinate \hat{x}_i and in general $[\hat{W}_i, \hat{W}_j] \neq 0$. Note that both Q_1 and Q_2 are constructed to be hermitian operators for real \hat{W}_i . The supercharges Q_1 and Q_2 satisfy the following standard superalgebra:

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha,\beta} H, \quad [H, Q_\alpha] = 0, \quad \alpha, \beta = 1, 2, \quad (8)$$

where the Hamiltonian H is given by

$$H = \frac{1}{2} \sum_{i=1}^N \left(\hat{p}_i^2 + \hat{W}_i^2 \right)$$

$$\begin{aligned}
 & -\frac{i}{4} \sum_{i,j=1}^N \left(B_{ij} \xi_i \xi_j + 2\xi_i \xi_{N+j} [\hat{p}_i, \hat{W}_j] \right. \\
 & \left. + i \xi_{N+i} \xi_{N+j} [\hat{W}_i, \hat{W}_j] \right). \tag{9}
 \end{aligned}$$

The Hamiltonian H is hermitian, since it is given by the square of the hermitian operator $Q_1(Q_2)$. The hermiticity of H can also be checked explicitly using (4). The term containing B_{ij} arises due to the non-commutativity among momentum operators. Similarly, the last term in H that is proportional to $[\hat{W}_i, \hat{W}_j]$ arises due to the non-commutativity among space coordinates. Such a term is absent for supersymmetric quantum mechanics on commutative space.

A few comments are in order at this point.

(i) If we allow θ_{ij}, B_{ij} and C_{ij} to be functions of the non-commutative coordinates \hat{x}_i, \hat{p}_i instead of c-number matrices, the whole analysis up to the construction of the super-Hamiltonian (9) remains valid. It is worth mentioning here that the Jacobi identities severely restrict the choice of the operators $\theta_{ij}, B_{ij}, C_{ij}$ for such more general theories.

(ii) In the standard construction of supersymmetric quantum mechanics on the commutative space, one usually introduces fermionic variables ψ_i and their conjugates ψ_i^\dagger ,

$$\psi_i = \frac{i}{2} (\xi_i - \xi_{N+i}), \quad \psi_i^\dagger = \frac{i}{2} (\xi_i + \xi_{N+i}), \tag{10}$$

so that the eigenstates can be labeled in terms of the total fermion number $N_F = \sum_i \psi_i^\dagger \psi_i$. However, one can check that H contains a term of the form $\psi_i \psi_j, \psi_i^\dagger \psi_j^\dagger$ for $B_{ij}, \theta_{ij} \neq 0$, implying that N_f is not a conserved quantity.

(iii) Let us define another set of supercharges q_1 and q_2 ,

$$q_1 = -\frac{1}{\sqrt{2}} \sum_{i=1}^N \left(\xi_{N+i} \hat{p}_i + i \xi_i \hat{W}_i \right), \quad q_2 = -i\gamma_5 q_1. \tag{11}$$

These two supercharges satisfy the standard superalgebra (8) with the Hamiltonian $h \equiv q_1^2 = q_2^2$,

$$\begin{aligned}
 h = & \frac{1}{2} \sum_{i=1}^N \left(\hat{p}_i^2 + \hat{W}_i^2 \right) \\
 & + \frac{i}{4} \sum_{i,j=1}^N \left(B_{ij} \xi_{N+i} \xi_{N+j} - 2\xi_i \xi_{N+j} [\hat{p}_j, \hat{W}_i] \right. \\
 & \left. + i \xi_i \xi_j [\hat{W}_i, \hat{W}_j] \right). \tag{12}
 \end{aligned}$$

In the commutative limit $B_{ij} = \theta_{ij} = C_{ij} = 0$, the two Hamiltonian H and h are identical. Moreover, in the same limit, the pair of charges (Q_1, q_1) and the pair (Q_2, q_2) satisfy the superalgebra (8) separately. Although (Q_1, Q_2, q_1, q_2) do not close under an enlarged $\mathcal{N} = 4$ superalgebra, we have the freedom of choosing any pair of the supercharges $\{(Q_1, Q_2), (q_1, q_2), (Q_1, q_1), (Q_2, q_2)\}$ for a given $\mathcal{N} = 2$ super-Hamiltonian $H = h$ on commutative space. However, if we take any of the parameters

$\theta_{ij}, B_{ij}, C_{ij}$ non-zero, such a freedom is completely lost. We are compelled to choose either the pair (Q_1, Q_2) or (q_1, q_2) and of course, in general, the super-Hamiltonian H and h are not identical.

2.1 Superoscillators

We will be working with Q_1, Q_2 and H in the rest of this paper. We also consider $\theta_{ij}, B_{ij}, C_{ij}$ as constant matrices from now onward. For the case of a superoscillator, we choose

$$\hat{W}_i = \omega \hat{x}_i. \tag{13}$$

The Hamiltonian now reads

$$\begin{aligned}
 H = & H_b - H_f \\
 H_b \equiv & \frac{1}{2} \sum_{i=1}^N (\hat{p}_i^2 + \omega^2 \hat{x}_i^2), \\
 H_f \equiv & \frac{\omega}{2} \left(\sum_{i=1}^N \xi_i \xi_{N+i} - \sum_{i \neq j} \xi_{N+i} \xi_j C_{ij} \right) \\
 & + \frac{i}{4} \sum_{i,j=1}^N (B_{ij} \xi_i \xi_j - \omega^2 \theta_{ij} \xi_{N+i} \xi_{N+j}). \tag{14}
 \end{aligned}$$

In the limit of $B_{ij}, \theta_{ij}, C_{ij} \rightarrow 0$, the Hamiltonian of the superoscillator on commutative space is recovered. Note that H_b is a function of the non-commutative co-ordinates and the momenta only, whereas H_f is solely expressed in terms of the elements of the Clifford algebra. This implies that H_b and H_f can be diagonalized separately.

A comment is in order at this point. The N -dimensional superoscillator is described in terms of $2N$ elements of the Clifford algebra (2). So, the Hamiltonian H_f can be expressed in terms of the linear combination of the $N(2N - 1)$ generators $\Sigma_{ij}^{1,2,3}$,

$$\begin{aligned}
 \Sigma_{ij}^1 = & \frac{i}{4} [\xi_i, \xi_j], \\
 \Sigma_{ij}^2 = & \frac{i}{4} [\xi_{N+i}, \xi_{N+j}], \\
 \Sigma_{ij}^3 = & \frac{i}{4} [\xi_i, \xi_{N+j}], \tag{15}
 \end{aligned}$$

of the group $SO(N, N)$ of rank N . Thus, in general, the eigenvalues of H_f can be expressed in terms of N quantum numbers. Further, in the matrix representation of the Clifford algebra (2), both the generators $\Sigma_{ij}^{1,2,3}$ and the Hamiltonian H_f can be expressed in terms of $2^N \times 2^N$ -dimensional matrices. Thus, each of the N quantum numbers can take only two values.

3 Two-dimensional superoscillator

We now specialize in this section to the non-commutative plane for which the antisymmetric matrices B_{ij} and θ_{ij}

can be parametrized in terms of single parameters B and θ , respectively. In particular,

$$C_{12} \equiv \phi_1, \quad C_{21} \equiv -\phi_2, \quad B_{ij} \equiv \epsilon_{ij}B, \quad \theta_{ij} \equiv \epsilon_{ij}\theta, \\ i, j = 1, 2. \quad (16)$$

With this choice for B_{ij} and θ_{ij} , there are many physically equivalent representations [14, 19] of the algebra (1) in terms of commutative canonically conjugate variables x_i and p_i satisfying

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\delta_{ij}. \quad (17)$$

As shown in [14, 15, 19], the Hamiltonian H_b with $N = 2$ can be equivalently written as a two-dimensional anisotropic oscillator. In particular,

$$H_b = \frac{1}{2} [\Omega_+ (p_1^2 + x_1^2) + \Omega_- (p_2^2 + x_2^2)],$$

$$2\Omega_{\pm} = \sqrt{(\omega^2\theta - B)^2 + 4\omega^2 + \omega^2(\phi_1 + \phi_2)^2} \\ \pm \sqrt{(\omega^2\theta + B)^2 + \omega^2(\phi_1 - \phi_2)^2}, \quad \kappa > 0,$$

$$2\Omega_{\pm} = \pm \sqrt{(\omega^2\theta - B)^2 + 4\omega^2 + \omega^2(\phi_1 + \phi_2)^2} \\ + \sqrt{(\omega^2\theta + B)^2 + \omega^2(\phi_1 - \phi_2)^2}, \quad \kappa < 0, \quad (18)$$

$$\Omega_+ = \sqrt{(\omega^2\theta + B)^2 + \omega^2(\phi_1 - \phi_2)^2}, \quad \Omega_- = 0, \quad \kappa = 0,$$

where the positive and the negative values of the parameter $\kappa \equiv \omega^2(1 - B\theta + \phi_1\phi_2)$ correspond to two different phases of the non-commutative oscillators, the critical value being $\kappa = 0$. The energy eigenvalues E_b of H_b is

$$E_b = \left(n_+^b + \frac{1}{2}\right) \Omega_+ + \left(n_-^b + \frac{1}{2}\right) \Omega_-, \quad (19)$$

where the quantum numbers n_{\pm}^b can take any non-negative integer values.

In order to diagonalize H_f , we use the following matrix representation of the Clifford algebra:

$$\xi_1 = i\sigma_1 \otimes \sigma_2, \quad \xi_2 = i\sigma_2 \otimes \sigma_2, \quad \xi_3 = -\sigma_3 \otimes \sigma_2, \quad \xi_4 = I \otimes \sigma_3, \quad (20)$$

where $\sigma_{1,2,3}$ are the three Pauli matrices and I is a 2×2 identity matrix. The Hamiltonian H_f is a 4×4 matrix,

$$H_f = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \\ \alpha \equiv \begin{pmatrix} \frac{B}{2} & \frac{\omega}{2}(i + \phi_1) \\ \frac{\omega}{2}(-i + \phi_1) & -\frac{B}{2} \end{pmatrix}, \\ \beta \equiv \begin{pmatrix} -\frac{\omega^2\theta}{2} & \frac{\omega}{2}(i + \phi_2) \\ \frac{\omega}{2}(-i + \phi_2) & \frac{\omega^2\theta}{2} \end{pmatrix}. \quad (21)$$

The eigenvalues of H_f are

$$\left\{ -\frac{1}{2}(\Omega_+ - \Omega_-), \frac{1}{2}(\Omega_+ - \Omega_-), -\frac{1}{2}(\Omega_+ + \Omega_-), \right. \\ \left. \frac{1}{2}(\Omega_+ + \Omega_-) \right\}. \quad (22)$$

For $\Omega_- = 0$, there are only two independent eigenvalues $\pm \frac{1}{2}\Omega_+$, each of them being doubly degenerate. The frequency Ω_- vanishes only in the critical phase $\kappa = 0$. Separate discussions are needed for the critical and the non-critical phases.

3.1 Critical phase $\kappa = 0$

The frequency $\Omega_- = 0$ in the critical phase. The eigenvalues E_f of H_f can be written in a closed form as

$$-E_f = \frac{1}{2}n_+^f n_-^f \Omega_+, \quad n_{\pm}^f = -1, 1. \quad (23)$$

The non-linear dependence on the ‘‘fermionic’’ quantum numbers n_{\pm}^f is due to the doubly degenerate eigenvalues $\pm \frac{1}{2}\Omega_+$. Combining (19) and (23), we find the energy eigenvalues E of the super-Hamiltonian H ,

$$E = E_b - E_f \\ = \left(n_+^b + \frac{1}{2}n_+^f n_-^f + \frac{1}{2}\right) \Omega_+, \quad (24)$$

where the quantum numbers n_{\pm}^b can take any non-negative integer values, while the ‘‘fermionic’’ quantum numbers n_{\pm}^f can take only two values, either -1 or 1 .

For $\Omega_- = 0$, a reduction in the phase space occurs. The phase-space variables p_2 and x_2 completely decouple from H_b . The energy eigenvalues E_b of the bosonic sector and consequently E becomes infinitely degenerate. Similarly, the four eigenvalues of H_f reduces to two with each of them being doubly degenerate. The ground-state energy $E = 0$ is obtained, either for

- (i) $n_+^b = 0, n_+^f = 1, n_-^f = -1$ or
- (ii) $n_+^b = 0, n_+^f = -1, n_-^f = 1$. The supersymmetry is unbroken.

The condition $\kappa = 0$ has two solutions,

- (a) $\omega = 0$ and
- (b) $1 - B\theta + \phi_1\phi_2 = 0$. In the limit $\omega \rightarrow 0$, $\Omega_+ = B$, and the eigenvalue equation of H reduces to that of Pauli equation on the two-dimensional commutative space. Note that the Pauli Hamiltonian in our case has higher symmetry than the one usually considered in the literature [27], since we have taken a 4×4 -dimensional matrix representation of the Clifford algebra instead of the familiar 2×2 -dimensional representation. So, the degeneracy structure of the energy spectrum is different from [27].

3.2 Non-critical phase $\kappa \neq 0$

In the non-critical phase, both $\Omega_{\pm} \neq 0$. The eigenvalue E_f ,

$$-E_f = \frac{1}{2} \left(n_-^f \Omega_- + n_+^f \Omega_+ \right), \quad (25)$$

is identical to that of an anisotropic “fermionic” oscillator with two degrees of freedom. The eigenvalue E of the super-Hamiltonian H is,

$$E = \left(n_+^b + \frac{1}{2}n_+^f + \frac{1}{2} \right) \Omega_+ + \left(n_-^b + \frac{1}{2}n_-^f + \frac{1}{2} \right) \Omega_- . \tag{26}$$

The spectrum E in (26) is identical to that of an anisotropic superoscillator on the commutative plane. The ground-state energy $E = 0$ is obtained for $n_{\pm}^b = 0, n_{\pm}^f = -1$. All excited states are paired together. The supersymmetry is unbroken.

The isotropic superoscillator can be obtained as a special case, when the parameters $B, \theta, \omega, \phi_{1,2}$ satisfy the following two relations simultaneously:

$$\omega^2\theta + B = 0, \quad \phi_1 = \phi_2, \quad \kappa > 0. \tag{27}$$

For $\kappa < 0$, there is no “isotropic limit”. The commutative limit, $B, \theta, \phi_{1,2} \rightarrow 0$ is of course a solution of the above equation (27) for which $\Omega_{\pm} = \omega^2$. Note that the isotropic superoscillator can be obtained as a special case only when both θ and B are either zero or non-zero simultaneously.

3.3 Inverted superoscillator on the non-commutative plane

In the commutative space, the inverted harmonic oscillator

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^2 (p_i^2 - \bar{\omega}^2 x_i^2), \tag{28}$$

with $\bar{\omega}$ real, is unbounded from below and there is no well-defined ground state. However, for the non-commutative case with both B and θ being non-zero, \bar{H}_b admits well-defined ground state [20]. This result can be understood intuitively as follows. The fundamental uncertainty relations are modified due to the non-commutativity and lower bounds appearing in these relations are expected to explicitly depend on θ and B . These parameters act as a kind of regulators/cut-offs to the singular potential for some ranges of their allowed values. Consequently, the potential effectively becomes bounded from below and admits well-defined ground state.

To analyze a similar situation in case of the supersymmetric inverted oscillator, we make the transformation $\omega \rightarrow i\bar{\omega}$ in all previous calculations in this section and identify any operator/function A undergoing such a transformation as \bar{A} . The transformation $\omega \rightarrow i\bar{\omega}$ amounts to taking a complex superpotential. The supercharges \bar{Q}_1, \bar{Q}_2 and the Hamiltonian \bar{H}_f are no more hermitian operators. Further, unlike the standard supersymmetric theory, the supersymmetric Hamiltonian \bar{H} is not a semi-positive definite quantity. These features are common to inverted superoscillator both on the commutative as well as on the non-commutative space. The inverted superoscillator on the commutative space does not have a well-defined ground state. However, in spite of the undesirable features like non-hermiticity and unboundedness from below, we

will show that the spectrum of the inverted superoscillator on the non-commutative space is indeed real. It is known that a certain class of non-hermitian Hamiltonians on ordinary commutative space admits real spectra [28, 29]. To the best of our knowledge, the discussion in this section is the first example of a non-hermitian Hamiltonian admitting real spectra in the context of a supersymmetric theory as well as on a non-commutative space.

We find the frequencies of \bar{H}_b ,

$$\begin{aligned} 2\bar{\Omega}_{\pm} &= \sqrt{(\bar{\omega}^2\theta - B)^2 - \bar{\omega}^2(\phi_1 - \phi_2)^2 + 4\bar{\kappa}} \\ &\quad \pm \sqrt{(\bar{\omega}^2\theta - B)^2 - \bar{\omega}^2(\phi_1 - \phi_2)^2}, \quad \bar{\kappa} > 0, \\ 2\bar{\Omega}_{\pm} &= \pm \sqrt{(\bar{\omega}^2\theta - B)^2 - \bar{\omega}^2(\phi_1 - \phi_2)^2 + 4\bar{\kappa}} \\ &\quad + \sqrt{(\bar{\omega}^2\theta - B)^2 - \bar{\omega}^2(\phi_1 - \phi_2)^2}, \quad \bar{\kappa} < 0, \\ \bar{\Omega}_+ &\equiv \Omega = \sqrt{(\bar{\omega}^2\theta - B)^2 - \bar{\omega}^2(\phi_1 - \phi_2)^2}, \\ \Omega_- &= 0, \quad \bar{\kappa} = 0, \end{aligned} \tag{29}$$

where $\bar{\kappa} = -\bar{\omega}^2(1 - B\theta + \phi_1\phi_2)$. We discuss the three cases separately.

(i) $\bar{\kappa} < 0$: The commutative limit, $(\phi_{1,2}, \theta, B) \rightarrow 0$, can be taken only in the phase $\bar{\kappa} < 0$. We find a complex $\bar{\Omega}_{\pm}$ in this limit, implying that the solutions are not stable. The commutative limit with non-zero magnetic field, i.e. $(\phi_{1,2}, \theta) \rightarrow 0$ and $B \neq 0$, also belongs to this phase. The frequencies $\bar{\Omega}_{\pm}$ are complex for $B^2 < 4|\bar{\kappa}|$ and Ω_- is negative for $B^2 > 4|\bar{\kappa}|$ signifying physically non-acceptable non-normalizable solutions. In fact, in the phase $\bar{\kappa} < 0$, there are no physically acceptable solutions, since $\bar{\Omega}_{\pm}$ are either complex or negative.

(ii) $\bar{\kappa} > 0$: There are regions in the parameter space for which physically acceptable solutions with well-defined ground-state energy can be found. In particular, for $(\bar{\omega}^2\theta - B)^2 \geq \bar{\omega}^2(\phi_1 - \phi_2)^2$, both $\bar{\Omega}_{\pm}$ are real and positive. When the bound is saturated, we get an isotropic oscillator with $\bar{\Omega}_+ = \bar{\Omega}_- = \sqrt{\bar{\kappa}}$. The energy spectrum is given by $\bar{E} = \left(n_-^b + \frac{1}{2}n_-^f + \frac{1}{2} \right) \bar{\Omega}_- + \left(n_+^b + \frac{1}{2}n_+^f + \frac{1}{2} \right) \bar{\Omega}_+$ with the zero ground-state energy.

(iii) $\bar{\kappa} = 0$: At the critical point $\bar{\kappa} = 0$ there is a reduction in the phase space. For $(\bar{\omega}^2\theta - B)^2 > \bar{\omega}^2(\phi_1 - \phi_2)^2$, Ω is real and positive. The energy spectrum is $\bar{E} = \left(n_+^b + \frac{1}{2}n_+^f + \frac{1}{2} \right) \Omega$ with a well-defined ground state $\bar{E}_0 = 0$. The energy spectrum is infinitely degenerate.

4 Three-dimensional superoscillator

In this section, we discuss the three-dimensional superoscillator on the non-commutative space. For simplicity, we choose $C_{ij} = 0 \forall i$ and j so that the canonical commutation relations between the non-commutative coordinates and the momenta now read $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$. Each of the antisymmetric matrices B_{ij} and θ_{ij} is parameterized in terms of three parameters. In particular,

$$B_{ij} \equiv \epsilon_{ijk} B_k, \quad \theta_{ij} \equiv \epsilon_{ijk} \theta_k. \tag{30}$$

We also define,

$$B \equiv \sqrt{B_1^2 + B_2^2 + B_3^2}, \quad \theta \equiv \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}, \quad (31)$$

for convenience. The bosonic part of the Hamiltonian is still given by H_b in (14) with $N = 3$. As is well known [15, 19] and described in Appendix A, the Hamiltonian H_b can be expressed solely in terms of canonically conjugate variables x_i and p_i as an anisotropic oscillator in three-dimensional commutative space. In particular,

$$H_b = \frac{1}{2} [\Omega_+ (p_1^2 + x_1^2) + \Omega_- (p_2^2 + x_2^2) + \Omega_0 (p_3^2 + x_3^2)], \quad (32)$$

where $\pm\Omega_{\pm}, \pm\Omega_0$ are the six eigenvalues of the 6×6 matrix iM given in appendix A with $C_{ij} = 0 \forall i$ and j . There are two phases of the theory characterized by $\kappa > 0$ or $\kappa < 0$ with $\kappa = 0$ being the critical phase, where $\kappa \equiv \Omega_+ \Omega_- \Omega_0$. The frequencies Ω_{\pm}, Ω_0 are always positive for the range of parameters that is determined depending on the particular phase ($\kappa = 0, \kappa > 0, \kappa < 0$) in which the superoscillators are being considered.

The Hamiltonian H_f with the parametrization (30) now reads

$$H_f = \frac{\omega}{2} \sum_{i=1}^3 \xi_i \xi_{3+i} + \frac{i}{4} \sum_{i,j=1}^3 \epsilon_{ijk} (B_k \xi_i \xi_j - \omega^2 \theta_k \xi_{3+i} \xi_{3+j}). \quad (33)$$

We choose the matrix representation of the Clifford algebra (2)

$$\begin{aligned} \xi_1 &= i\sigma_3 \otimes \sigma_2 \otimes I, & \xi_2 &= i\sigma_1 \otimes \sigma_2 \otimes I, \\ \xi_3 &= i\sigma_2 \otimes I \otimes \sigma_3, \\ \xi_4 &= \sigma_2 \otimes I \otimes \sigma_1, & \xi_5 &= I \otimes \sigma_3 \otimes \sigma_2, \\ \xi_6 &= I \otimes \sigma_1 \otimes \sigma_2. \end{aligned} \quad (34)$$

The Hamiltonian H_f in terms of these matrix representation of the elements of the Clifford algebra is given in Appendix B.

A general analysis of the spectrum of H involves the diagonalization of the 6×6 matrix M and 8×8 matrix H_f . The eigenvalues of both of these matrices come into positive-negative pairs. Thus, all the eigenvalues of H can be obtained analytically by solving a cubic and a quartic characteristic equation, respectively. However, the final expressions for the frequencies Ω_i are quite messy and complicated. We present here results for a few specific choices of parameters with $\omega > 0$. The results for $\omega < 0$ can be obtained trivially by considering the negative pairs of the frequencies Ω_{\pm}, Ω_0 . The results for the purely bosonic model has been obtained previously for some of the cases considered below, while the results for the fermionic part and hence the complete supersymmetric model are obtained for the first time in this paper. We would like to mention here that the supersymmetry is unbroken for all the cases considered below.

(a) $\mathbf{B}_i = \mathbf{0}, \forall i$: The non-commutativities among the momentum operators are switched off. There is only one

phase characterized by $\kappa = \omega^3 > 0$ for positive ω . The frequencies are

$$\Omega_0 = \omega, \quad \Omega_{\pm} = \omega \sqrt{t \pm \sqrt{t^2 - 1}}, \quad t \equiv 1 + \frac{1}{2} \omega^2 \theta^2. \quad (35)$$

There is no limit for non-zero θ_i 's for which $\Omega_+ = \Omega_-$. The eight eigenvalues of H_f ,

$$\left\{ \frac{\omega}{2} (1 \pm \sqrt{2+2t}), \quad -\frac{\omega}{2} (1 \pm \sqrt{2+2t}), \quad (36) \right. \\ \left. \frac{\omega}{2} (1 \pm \sqrt{2t-1}), \quad -\frac{\omega}{2} (1 \pm \sqrt{2t-1}) \right\},$$

can be written in a closed form through the introduction of the three quantum numbers n_{\pm}^f, n_0^f ,

$$-E_f = \frac{1}{2} (n_-^f \Omega_- + n_+^f \Omega_+ + n_0^f \Omega_0), \quad n_{\pm}^f, n_0^f = -1, 1. \quad (37)$$

Thus, the eigenspectrum of H ,

$$E = \left(n_+^b + \frac{1}{2} n_+^f + \frac{1}{2} \right) \Omega_+ + \left(n_-^b + \frac{1}{2} n_-^f + \frac{1}{2} \right) \Omega_- \\ + \left(n_0^b + \frac{1}{2} n_0^f + \frac{1}{2} \right) \Omega_0, \quad (38)$$

is that of an anisotropic superoscillator on the three-dimensional commutative space. The bosonic quantum numbers n_{\pm}^b, n_0^b can take any non-negative integer values. The supersymmetric ground state $E = 0$ is obtained for $n_{\pm}^b = n_0^b = 0, n_{\pm}^f = n_0^f = -1$.

(b) $\mathbf{B}_i = -\omega^2 \theta_i, \forall i$: For this choice, two of the three frequencies become equal,

$$\Omega_0 = \omega, \quad \Omega_+ = \Omega_- \equiv \Omega = \omega \sqrt{1 + \omega^2 \theta^2}, \quad (39)$$

producing an isotropic oscillator on the x - y plane plus an oscillator along the z -direction. There is again only one phase $\kappa > 0$ for positive ω . The eight eigenvalues of H_f can be expressed in a closed form as in (37) with the frequencies given by (39). The eigenspectrum of H ,

$$E = \left(n_+^b + n_-^b + \frac{1}{2} n_+^f + \frac{1}{2} n_-^f + 1 \right) \Omega \\ + \left(n_0^b + \frac{1}{2} n_0^f + \frac{1}{2} \right) \Omega_0, \quad (40)$$

has a higher level of degeneracy compared to (38). The supersymmetric ground state is obtained for all bosonic quantum numbers being zero and all the fermionic quantum numbers equal to -1 .

(c) $\mathbf{B}_i = \omega^2 \theta_i, \forall i$: There are two phases in the theory characterized by $\kappa = \omega^3 (1 - \omega^2 \theta^2)$. The frequencies in these two phases are

$$\Omega_0 = \omega, \quad \Omega_{\pm} = \omega (1 \pm \omega \theta), \quad \kappa > 0, \\ \Omega_0 = \omega, \quad \Omega_{\pm} = \omega (\pm 1 + \omega \theta), \quad \kappa < 0. \quad (41)$$

There is no isotropic point $\Omega_- = \Omega_+$ for $\theta_i \neq 0$. The eigenspectrum of H is given by (38) with the frequencies are as in (41).

(d) Critical point $\kappa = 0$: The critical point $\kappa = 0$ separating the two phases of the superoscillators is determined from the relation $\text{Det}(M) = \omega^6 \left(1 - \sum_{i=1}^3 \theta_i B_i\right) = 0$.

There are two cases, (i) $\sum_{i=1}^3 \theta_i B_i = 1$ and (ii) $\omega = 0$.

(i) $\sum_{i=1}^3 \theta_i B_i = 1$: Without any loss of generality, we choose a particular solution, $B_i = \theta_i \theta^{-2}$. The phase space is reduced to 4 dimensions, since one of the frequencies vanishes. In particular,

$$\Omega_0 = 0, \quad \Omega_+ = \omega, \quad \Omega_- = \theta^{-1} (1 + \omega^2 \theta^2). \quad (42)$$

The bosonic spectrum becomes infinitely degenerate. At the critical point, there are four independent eigenvalues of H_f with each of them being doubly degenerate,

$$\left\{ \pm \frac{\omega}{2} + \frac{1}{2\theta} (1 + \theta^2 \omega^2), \pm \frac{\omega}{2} - \frac{1}{2\theta} (1 + \theta^2 \omega^2) \right\}. \quad (43)$$

The eigenvalues of H can be written in a closed form as

$$E = \left(n_+^b + \frac{1}{2} n_0^f n_+^f + \frac{1}{2} \right) \Omega_+ + \left(n_-^b + \frac{1}{2} n_0^f n_-^f + \frac{1}{2} \right) \Omega_-. \quad (44)$$

The $E = 0$ ground state is obtained for

- (i) $n_+^b = 0, n_0^f = 1, n_{\pm}^f = -1$ and
- (ii) $n_+^b = 0, n_0^f = -1, n_{\pm}^f = 1$. The whole spectrum including the ground state is of course infinitely degenerate due to the reduction in the phase space.
- (ii) $\omega = 0$: The eigenvalue equation of the Hamiltonian H is that of a three-dimensional Pauli equation. There are further reductions in the phase space, compared to the case described above. Four out of the six phase-space variables decouple from the dynamics. In particular,

$$\Omega_{\pm} = 0, \quad \Omega_0 = B. \quad (45)$$

The eigenvalues of H_f are $\pm \frac{1}{2} \Omega_0$ with each of them having four-fold degeneracy. The eigenspectrum of H can be written as

$$E = \left(n_0^b + \frac{1}{2} n_0^f n_-^f n_+^f + \frac{1}{2} \right) \Omega_0. \quad (46)$$

The $E = 0$ ground state can be obtained for

- (i) $n_0^b = 0, n_0^f = -1, n_{\pm}^f = 1$,
- (ii) $n_0^b = 0, n_0^f = -1, n_{\pm}^f = -1$,
- (iii) $n_0^b = 0, n_0^f = 1, n_{\pm}^f = \pm 1$ and
- (iv) $n_0^b = 0, n_0^f = 1, n_{\pm}^f = \mp 1$. The whole spectrum including the ground state is of course infinitely degenerate due to the reduction in the phase space.

5 Summary and discussion

We have constructed a supersymmetric quantum mechanics on non-commutative space in terms of $\mathcal{N} = 2$ real supercharges. This construction is valid in any arbitrary dimensions for arbitrary superpotential. Further,

the same construction is valid for a very large class of non-commutativities among phase-space operators. The non-commutativity among the space coordinates and/or among momentum coordinates restricts the number of independent ways one can construct $\mathcal{N} = 2$ supersymmetric theory for a given superpotential.

We have studied the non-commutative superoscillators in two and three dimensions in some detail and obtained their eigenspectra analytically. The spectrum of the non-commutative superoscillator in two (three) dimensions is identical to an anisotropic oscillator in the commutative two (three) space dimensions. There is a critical surface in the parameter space for which there is reduction in the phase space. Consequently, the spectrum due to the bosonic sector becomes infinitely degenerate, while the degeneracy in the fermionic sector is doubled. This is because the bosonic quantum numbers can take any non-negative integer values, while the fermionic quantum numbers can take only two values.

We have studied the ‘‘inverted superoscillator’’ on the non-commutative plane. The corresponding Hamiltonian is neither hermitian nor a semi-positive definite operator. In spite of these, we have found the spectrum to be real, positive and with a well-defined ground state for a few specific choices of the parameters. For these choices of parameters, there is no smooth limit to the ordinary commutative space. Thus, the energy spectrum with a well-defined ground state for an ‘‘inverted superoscillator’’ is purely the effect of non-commutativity.

Acknowledgements. It is a great pleasure to acknowledge several useful discussions with Prof. Avinash Khare. I also thank him for a very critical reading of the manuscript. This work is supported (DO No. SR/FTP/PS-06/2001) by SERC, DST, Govt. of India through the Fast Track Scheme for Young Scientists: 2001–2002.

Appendix A: Mapping of H_b to an anisotropic oscillator on commutative space for $N = 3$

In this appendix, we describe a general procedure for mapping the non-commutative Hamiltonian H_b to an equivalent Hamiltonian of anisotropic oscillators on the commutative space. The discussion is based on [19, 15, 30].

Define a six-dimensional vector $U = (\omega \hat{x}_1, \hat{p}_1, \omega \hat{x}_2, \hat{p}_2, \omega \hat{x}_3, \hat{p}_3)$ in the phase space. The algebra in (1) can be written as

$$[U_I, U_J] = iM_{IJ}, \quad I, J = 1, 2, \dots, 6, \quad (A.1)$$

where the 6×6 -dimensional matrix M is given by

$$M = \begin{pmatrix} 0 & \omega & \omega^2 \theta_3 & \omega C_{12} & -\omega^2 \theta_2 & \omega C_{13} \\ -\omega & 0 & -\omega C_{21} & B_3 & -\omega C_{31} & -B_2 \\ -\omega^2 \theta_3 & \omega C_{21} & 0 & \omega & \omega^2 \theta_1 & \omega C_{23} \\ -\omega C_{12} & -B_3 & -\omega & 0 & -\omega C_{32} & B_1 \\ \omega^2 \theta_2 & \omega C_{31} & -\omega^2 \theta_1 & \omega C_{32} & 0 & \omega \\ -\omega C_{13} & B_2 & -\omega C_{23} & -B_1 & -\omega & 0 \end{pmatrix}. \quad (A.2)$$

There exists an orthogonal transformation such that the matrix M can be block-diagonalized as [30]

$$R^T M R = \begin{pmatrix} D_1 & 0 & & \\ 0 & D_2 & 0 & \\ 0 & 0 & D_3 & \end{pmatrix}, \quad D_i \equiv \begin{pmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{pmatrix}, \quad (\text{A.3})$$

using a $O(6)$ rotational matrix R and its transpose R^T , where $\pm\Omega_i$ are the six eigenvalues of iM . The matrix R is unique modulo $O(6)$ rotations. The transformed variables in the phase space, $(x_1, p_1, x_2, p_2, x_3, p_3) \equiv \left(\frac{u_1}{\sqrt{|\Omega_1|}}, \frac{u_2}{\sqrt{|\Omega_1|}}, \frac{u_3}{\sqrt{|\Omega_2|}}, \frac{u_4}{\sqrt{|\Omega_2|}}, \frac{u_5}{\sqrt{|\Omega_3|}}, \frac{u_6}{\sqrt{|\Omega_3|}} \right)$ with $u = R^T U$ satisfy the usual canonical commutation relations

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = i\delta_{ij}. \quad (\text{A.4})$$

The Hamiltonian is $H_b = \frac{1}{2} \sum_{I=1}^6 U_I^2 = \frac{1}{2} \sum_{I=1}^6 u_I^2$, because of the $O(6)$ invariance. We get back the Hamiltonian (32) once we express the variables u_I in terms of x_i and p_i .

For $N = 2$, We can truncate the matrix M to a 4×4 matrix by taking the first four row and columns only, i.e. M_{IJ} with $I, J = 1, 2, 3, 4$. Identifying $\theta_3 = \theta$, $B_3 = B$, $C_{12} = \phi_1$ and $c_{21} = -\phi_2$, all the relevant results of Sect. 3 can be obtained.

Appendix B: H_f in matrix representation for $N = 3$

The Hamiltonian H_f in terms of the matrix representation of the Clifford algebra has the following form:

$$H_f = \begin{pmatrix} \Gamma_1 & \eta \\ \eta^\dagger & \Gamma_2 \end{pmatrix}, \quad (\text{B.1})$$

where the 4×4 matrices Γ_1 , Γ_2 and η are

$$\begin{aligned} \Gamma_1 &\equiv \begin{pmatrix} 0 & -\gamma_3^+ & -\gamma_1^+ & -\gamma_2^- \\ \gamma_3^+ & 0 & -\gamma_2^+ & \gamma_1^- \\ \gamma_1^+ & \gamma_2^+ & 0 & -\gamma_3^- \\ \gamma_2^- & -\gamma_1^- & \gamma_3^- & 0 \end{pmatrix}, \\ \Gamma_2 &\equiv \begin{pmatrix} 0 & -\gamma_3^- & \gamma_1^- & \gamma_2^- \\ \gamma_3^- & 0 & \gamma_2^+ & -\gamma_1^+ \\ -\gamma_1^- & -\gamma_2^+ & 0 & -\gamma_3^+ \\ -\gamma_2^- & \gamma_1^+ & \gamma_3^+ & 0 \end{pmatrix}, \\ \eta &\equiv \begin{pmatrix} 0 & 0 & 0 & -\frac{i}{2}\omega \\ 0 & 0 & \frac{i}{2}\omega & 0 \\ 0 & \frac{i}{2}\omega & 0 & 0 \\ \frac{3i}{2}\omega & 0 & 0 & 0 \end{pmatrix}, \\ \gamma_i^\pm &\equiv \frac{i}{2}(B_i \pm \theta_i \omega^2). \end{aligned} \quad (\text{B.2})$$

Note that

$$(\gamma_i^\pm)^* = -\gamma_i^\pm, \quad (\text{B.3})$$

where $(X)^*$ denotes the complex conjugate of X . The hermiticity of H_f can now be checked easily.

References

1. A. Connes, Non-commutative geometry (Academic Press, London 1994)
2. N. Seiberg, E. Witten, JHEP **9909**, 032 (1999) [hep-th/9908142]
3. S. Minwalla, M.V. Raamsdonk, N. Seiberg, JHEP **0002**, 020 (2000) [hep-th/9912072]
4. R. Gopakumar, S. Minwalla, A. Strominger, JHEP **0005**, 020 (2000) [hep-th/0003160]
5. M.R. Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001) [hep-th/0106048]
6. J.A. Harvey, Komaba Lectures on Non-commutative Solitons and D-Branes, hep-th/0102076
7. R.J. Szabo, Int. J. Mod. Phys. A **19**, 1837 (2004) [physics/0401142]; Phys. Rep. **378**, 207 (2003) [hep-th/0109162]
8. V.O. Rivelles, hep-th/0305122
9. D. Klemm, S. Penati, L. Tamassia, Class. Quant. Grav. **20**, 2905 (2003) [hep-th/0104190]
10. N. Seiberg, JHEP **0306**, 010 (2003) [hep-th/0305248]
11. C. Duval, P. Horvathy, Phys. Lett. B **479**, 284 (2000) [hep-th/0002233]; J. Phys. A **34**, 10097 (2001) [hep-th/0106089]; P. Horvathy, Ann. Phys. **299**, 128 (2002) [hep-th/0201007]
12. M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001) [hep-th/0010175]
13. M. Chaichian, P. Presnajder, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Lett. B **527**, 149 (2002) [hep-th/0012175]
14. V.P. Nair, A.P. Polychronakos, Phys. Lett. B **505**, 267 (2001) [hep-th/0011172]
15. A. Hatzinikitas, I. Smyrnakis, J. Math. Phys. **43**, 113 (2002) [hep-th/0103074]
16. A. Jellal, J. Phys. A **34**, 10159 (2001) [hep-th/0105303]; S. Bellucci, A. Nersessian, C. Sochichiu, Phys. Lett. B **52**, 345 (2001) [hep-th/0106138]; A. Smalagic, E. Spallucci, Phys. Rev. D **65**, 107701 (2002) [hep-th/0108216]; B. Muthukumar, P. Mitra, Phys. Rev. D **66**, 027701 (2002) [hep-th/0204149]
17. R. Banerjee, Mod. Phys. Lett. A **17**, 631 (2002) [0106280]
18. A. Smalagic, E. Spallucci, J. Phys. A **365**, L363 (2002) [hep-th/0205242]
19. L. Jonke, S. Meljanac, Eur. Phys. J. C **29**, 433 (2003) [hep-th/0210042]; I. Dacic, L. Jonke, S. Meljanac, hep-th/0301066
20. S. Bellucci, Phys. Rev. D **67**, 105014 (2003) [hep-th/0301227]
21. F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. **251**, 267 (1995)
22. T. Curtright, D. Fairlie, C.K. Zachos, Phys. Rev. D **58**, 025002 (1998) [hep-th/9711183]
23. E. Harikumar, V. Sunil Kumar, A. Khare, Phys. Lett. B **589**, 155 (2004) [hep-th/0402064]
24. P. Pouliot, Class. Quant. Grav. **21**, 145 (2004) [hep-th/0306261]

25. P.K. Ghosh, Nucl. Phys. B **681**, 359 (2004) [hep-th/0309183]
26. E. D'Hoker, L. Vinet, Comm. Math. Phys. **97**, 391 (1985)
27. M. de Crombrugghe, V. Rittenberg, Ann. Phys. **151**, 99 (1983); A. Khare, J. Maharana, Nucl. Phys. B **244**, 409 (1984)
28. C.M. Bender, D.C. Brody, H.F. Jones, Am. J. Phys. **71**, 1095 (2003) [hep-th/0303005]; C.M. Bender, S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998)
29. P.K. Ghosh, quant-ph/0501087
30. D. McDuff, D. Salamon, Introduction to symplectic topology (Oxford Science Publications 1998)